Spontaneous Symmetry Breaking and Field Theory
Elective course
SS 2015
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Exercise 1

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1. For an open Ising chain of N spins the partition function is given by

$$Z = 2^N \prod_{i=1}^{N-1} \cosh \beta J_i \quad , \quad \beta = \frac{1}{\mathrm{k}T} \, .$$

Thereby J_i denotes the exchange interaction parameter between the spins at sites i and $i+1$. We first assume that the values of the J_i are fixed ("quenched disorder") and distributed randomly according to the following distribution ($0 < z < 1$)

$$P(J_i) = \begin{cases} \frac{1}{2\,z\,J} & \text{for } J(1-z) \le J_i \le J(1+z) \\ 0 & \text{otherwise .} \end{cases}$$

Calculate the free energy and the specific heat for that case for $T \to 0$ and $T \to \infty$. Then we assume that the values of J_i from the range $J(1-z) \le J_i \le J(1+z)$ are also in thermal equilibrium ("annealed disorder") and calculate again the free energy and the specific heat for $T \to 0$ and $T \to \infty$.

2. We assume that the positions of the atoms of a onedimensional quasicrystal are given by

$$x_N = N + \alpha + \frac{1}{\rho} \left\lfloor \frac{N}{\sigma} + \beta \right\rfloor \, .$$

Here $\lfloor z \rfloor$ is the largest integer number smaller than z, $\sigma > 1$ is an arbitrary irrational number, and $\rho > 0$, α and β are real numbers.

a. What are the distances between neighbouring atoms, and what is the probability to find one of these distances?

b. Show that (apart from a re-labeling) the quasicrystal remains the same when substituting simultaneously α by $\alpha + p + \frac{q}{\rho}$ and β by $\beta - q + \frac{p}{\sigma}$ with integer numbers q and p.

c. It can be shown that the above defined positions define a quasiperiodic arrangement of long (L) and short (S) atom distances. For the case $\sigma = \tau = (1+\sqrt{5})/2$ this sequence can be generated also in another way, starting from a single segment (L or S), by the repeated application of the substitution rule $L \to L\,S$ (i.e., an interval L in the sequence is replaced by a pair of intervals L and S) and $S \to L$ (i.e., the interval S is replaced by L). What is the value of L/S (and correspondingly of ρ) for which the substitution rule generates a self-similarity transformation (i.e., a transformation which does not change the statistical properties of the quasicrystal)?

3. The Fibonacci numbers F_n are defined by

$$F_n = F_{n-1} + F_{n-2} \quad \text{with} \qquad F_0 = 0 \quad \text{and} \quad F_1 = 1\,. \tag{1}$$

a. Demonstrate that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1}{2}(1+\sqrt{5})\right)^n - \left(\frac{1}{2}(1-\sqrt{5})\right)^n\right]\,.$$

Hint: Use the ansatz $F_n = x^n$ in eq.(1)

b. Calculate $\lim_{n\to\infty} F_{n+1}/F_n$.

c. Figure out the interrelation between the substitution rule of exercise **2c** and the Fibonacci numbers. Use the result of **3b** to determine the probability to find intervals with length L and S.