
Problem Set 1

Problem 1 Operator Algebra (Homework)

(a) Show that the following relations are valid:

- (i) $(c\hat{A})^\dagger = c^* \hat{A}^\dagger$
- (ii) $(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$
- (iii) $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$
- (iv) $(\hat{A}^\dagger)^\dagger = \hat{A}$

(b) Prove the following commutator identities:

- (i) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$
- (ii) $[\hat{A}, \hat{B}]^\dagger = [\hat{B}^\dagger, \hat{A}^\dagger]$
- (iii) $[[\hat{A}, \hat{B}], \hat{C}] + [[\hat{B}, \hat{C}], \hat{A}] + [[\hat{C}, \hat{A}], \hat{B}] = 0$ (Jacobi identity)

(c) The exponent of an operator is defined by the power expansion

$$e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^n. \tag{1}$$

Using this definition, or otherwise, express the operator

$$\hat{S}(a) = e^{a\hat{A}}, \quad \text{where } \hat{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } a \in \mathbb{R}, \tag{2}$$

as a 2×2 matrix.

Problem 2 Practice with Postulates in Quantum Mechanics

Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by the three kets $|1\rangle, |2\rangle, |3\rangle$. In this basis, the Hamiltonian $\hat{H}$ of the system and the two observables $\hat{A}$ and $\hat{B}$ are written as

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \hat{A} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{B} = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

where ω, a and b are positive real constants.
The system is at time $t = 0$ in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{1}{2} |2\rangle + \frac{1}{2} |3\rangle. \tag{4}$$

(a) At time $t = 0$, the energy of the system is measured. What values can be found, and with what probabilities? Calculate for the system in the state $|\psi(0)\rangle$ the mean value $\langle\hat{H}\rangle$ and the root-mean-square deviation $\Delta\hat{H}$.

(b) Instead of measuring $\hat{H}$ at time $t = 0$ one measures $\hat{A}$. What results can be found and with what probabilities? What is the state vector immediately after the measurement?

(c) Calculate the state vector $|\psi(t)\rangle$ of the system at time t.

(d) Calculate the mean values $\langle\hat{A}\rangle(t)$ and $\langle\hat{B}\rangle(t)$ of $\hat{A}$ and $\hat{B}$ at time t. What comments can be made?

(e) What results are obtained if the observable $\hat{A}$ is measured at time t? Same question for the observable $\hat{B}$.

Problem 3 Free Propagator

The propagator $K(x, x', t)$ for the Hamiltonian $\hat{H}$ is defined through the solution of the Schrödinger equation

$$[i\hbar\partial_t - \hat{H}(p, x)]K(x, x', t) = 0, \tag{5}$$

with the initial condition $K(x, x', 0) = \delta(x - x')$.

(a) Show that for an arbitrary initial condition $\psi(x, t = 0)$ the solution of the Schrödinger equation is given by

$$\psi(x, t) = \int dx' K(x, x', t)\psi(x', 0). \tag{6}$$

(b) Show, by using Fourier transformation (plane wave expansion), that the propagator for free particles with $\hat{H} = \hat{p}^2/2m$ is given by

$$K(x, x', t) = \left(\frac{m}{2\pi\hbar it}\right)^{1/2} \exp\frac{im(x - x')^2}{2\hbar t}. \tag{7}$$

(c) Consider the following initial condition:

$$\psi(x, 0) = Ae^{ik_0x}e^{-\frac{(x-x_0)^2}{4\sigma}}. \tag{8}$$

(i) Calculate the norm A.

(ii) Compute $\psi(x, t)$ using equation (6)

(iii) Find the expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$.