Advanced Quantum Theory **WS 2015/16**

Problem Set 2
Marks 34 Due: Thu 29 October 2015

Problem 4 Harmonic Oscillator (Homework)

(3+5+4+2)=14

Consider the one dimensional harmonic oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{1}$$

(a) Rewrite the Hamiltonian in terms of the non-Hermitian operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \quad \text{and} \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right), \tag{2}$$

known as the **annihilation operator** and **creation operator**, respectively.

(b) Calculate the following commutation relations:

$$[\hat{a}, \hat{a}^\dagger] \qquad [\hat{x}, \hat{a}^\dagger] \qquad [\hat{a}^m, \hat{a}^\dagger] \qquad [\hat{H}, \hat{x}] \qquad [\hat{H}, \hat{p}] \qquad \text{with } m \in \mathbb{N}. \tag{3}$$

(c) Find the ground state wave function $\psi_0(x)$. From that calculate the wave function $\psi_1(x)$ of the first excited state.

(d) In dimensionless units ($\hbar = \omega = m = 1$) one unnormalized energy eigenfunction is given by

$$\psi_a(x) = (2x^3 - 3x)e^{-x^2/2}. \tag{4}$$

Find two other (unnormalized) eigenfunctions which are closest in energy to $\psi_a(x)$.

Problem 5 Uncertainty Relation Oral

(2+2+3+4)=11

Let $\hat{A}$ and $\hat{B}$ be two Hermitian operators that do not commute. The **variance**[1] of an operator is defined by:

$$\Delta A = \sqrt{\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle}, \tag{5}$$

where $\langle ... \rangle$ is the expectation value in an arbitrary state. Then the following inequality holds:

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left|\langle[\hat{A}, \hat{B}]\rangle\right|^2. \tag{6}$$

This is the general formulation of Heisenberg's uncertainty relation.

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[1] Sometimes also called **uncertainty** or **root-mean-square deviation**.

(a) Show for the momentum and position operatore, the uncertainty relation

$$\Delta p . \Delta x \geq \frac{\hbar}{2} \tag{7}$$

(b) Prove **Schwarz's inequality**:

$$| \langle \psi | \phi \rangle |^2 \leq \langle \phi | \phi \rangle \langle \psi | \psi \rangle . \tag{8}$$

Hint: Decompose the state $|\psi\rangle$ into a parallel and a perpendicular component with respect to $|\phi\rangle$.

(c) Now derive eqn. (6) in the following way:

- Define the operators $\hat{A}' = \hat{A} - \langle \hat{A} \rangle$ and $\hat{B}' = \hat{B} - \langle \hat{B} \rangle$ and apply them to a general state $|\alpha\rangle$. Then use Schwarz's inequality.
- The product of two operators can be decomposed into two parts:

$$\hat{A}\hat{B} = \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{2}\{\hat{A}, \hat{B}\}, \tag{9}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denotes the **anticommutator**. Is the anticommutator Hermitian or anti-Hermitian? What follows for the expectation value of a Hermitian operator? Same question for anti-Hermitian operator.

Problem 6 Constant Potential in One Dimension (Oral)

(1+4+2)=7

Consider the following one dimensional potential:

$$\begin{aligned} V(x) &= \infty \quad x < 0 \\ V(x) &= -V_0 \quad 0 < x < a \\ V(x) &= 0 \quad x > a \end{aligned} \tag{10}$$

where V_0 is positive.

(a) Make a sketch of the potential.

(b) Find out the probability for transmission through the potential? For what value of E will this probability be unity?