Problem 10  Coherent States (Oral)

A coherent state of a one dimensional simple harmonic oscillator is defined to be an eigenstate of the non-Hermitian annihilation operator $\hat{a}$:

$$\hat{a} |\lambda\rangle = \lambda |\lambda\rangle$$  \hfill (1)

Where $\lambda$ is, in general a complex number.

(a) Prove that

$$|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger} |0\rangle$$  \hfill (2)

is a normalized coherent state.

(b) Write $|\lambda\rangle$ as

$$\lambda = \sum_{n=0}^{\infty} f(n) |n\rangle$$  \hfill (3)

Show that the distribution of $|f(n)|^2$ with respect to $n$ is of the Poisson form. Find the most probable value of $n$, hence of $E$.

(c) Show that a coherent state can also be obtained by applying the translation (finite displacement) operator $e^{ip\ell/\hbar}$ (where $p$ is the momentum operator and $\ell$ is the displacement) to the ground state.

(d) Show that the coherent state $\lambda$ remains coherent under time evolution and calculate the time-evolved state $|\lambda(t)\rangle$ (Hint: directly apply the time-evolution operator.)

Problem 11  Correlation Function (Oral)

Consider a function, known as correlation function defined by

$$C(t) = \langle x(t)x(0) \rangle$$  \hfill (4)

where $x(t)$ is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of a one dimensional simple harmonic oscillator.

Problem 12  Translation operator (Written)

Consider the translation operator defined by

$$\hat{T}(a) |x\rangle = |x + a\rangle ,$$  \hfill (5)

where $|x\rangle$ is the coordinate basis and $a$ a real constant.
(a) Convince yourself that when $\hat{T}(a)$ acts on a wave function we obtain
\[ \hat{T}(a)\psi(x) = \psi(x - a). \] 
(6)

(b) Show that the translation operator commutes with the momentum operator.

(c) Show that $\hat{T}(a)$ is a unitary operator, $\hat{T}^\dagger(a)\hat{T}(a) = \hat{1}$. What are its eigenvalues and associated eigenvectors? Further show that it is sufficient to know the wave function within an interval of length $a$ only.