**Problem Set 7** Due: 3/4 December 2015

## Problem 20 Spin-$\frac{1}{2}$–Spin-$\frac{1}{2}$ Coupling (Written)

A system of two spin-$\frac{1}{2}$ particles is described by the effective Hamiltonian

$$\hat{H} = A\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 + B(\hat{S}_{1z} + \hat{S}_{2z}), \tag{1}$$

where $A$ and $B$ are constants. Find all energy levels of this Hamiltonian.

## Problem 21 Positronium Atom (Oral)

The Hamiltonian of the positronium atom in the $1S$ state ($n = 1$, $l = 0$) in a magnetic field $\boldsymbol{B}$ along the $z$-axis is to a good approximation

$$\hat{H} = A\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 + \frac{eB}{mc}(\hat{S}_{1z} - \hat{S}_{2z}), \tag{2}$$

if all higher energy states are neglected. The electron is labeled as particle 1 and the positron as particle 2.

(a) Using the **coupled representation** $\{|S_1 S_2 S M\rangle\}$ in which $\hat{\boldsymbol{S}}^2 = (\hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2)^2$ and $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$ are diagonal, obtain the energy eigenvalues and eigenvectors of the Hamiltonian.

*Hint:*

*(i) One eigenvalue is* $E = \frac{1}{4}A\hbar^2$.

*(ii) You have to express either* $\hat{S}_{1z}$ *or* $\hat{S}_{2z}$ *in the basis* $\{|S_1 S_2 S M\rangle\}$. *To obtain the matrix elements in this basis, write* $\{|S_1 S_2 S M\rangle\}$ *in terms of the uncoupled basis* $\{|S_1 m_1 S_2 m_2\rangle\}$ *using the Clebsch-Gordan coefficients:*

$$|j_1 j_2 J M\rangle = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | j_1 j_2 J M\rangle \, |j_1 m_1 j_2 m_2\rangle \, . \tag{3}$$

(b) Now write the Hamiltonian in the **uncoupled representation** $\{|S_1 m_1 S_2 m_2\rangle \equiv |S_1 m_1\rangle \otimes |S_2 m_2\rangle\}$ and calculate the energy eigenvalues and eigenvectors.

*Note: The tensor product of two matrices* $\hat{A}$ *and* $\hat{B}$ *is given by*

$$\hat{A} \otimes \hat{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ & \\ a_{21} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} \tag{4}$$

### Problem 22 Clebsch-Gordan Coefficients (Written)

Calculate all the Clebsch-Gordan coefficients for $j_1 = 1$ and $j_2 = 1$.

### Problem 23 Density Operator (Oral)

In the basis where $\hat{\boldsymbol{\sigma}}_z$ is diagonal, three observables are given by

$$\hat{A} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} \tag{5}$$

The system is prepared in a spin state where the expectation values are found to be:

$$\langle \hat{A} \rangle = 2, \quad \langle \hat{B} \rangle = \frac{1}{2}, \quad \langle \hat{C} \rangle = 0. \tag{6}$$

(a) Determine the density matrix $\hat{\rho}$ of this spin state.

(b) Is the spin state a pure or mixed state?

(c) What is the probability of measuring $\frac{\hbar}{2}$ for the spin along the $z$-direction?

(d) Calculate the expectation values $\langle \hat{\sigma}_x \rangle$, $\langle \hat{\sigma}_y \rangle$, $\langle \hat{\sigma}_z \rangle$.