Problem Set 10 Due: 7/8 January 2016

Problem 29 Perturbation (Oral)

The Hamiltonian for an isotropic harmonic oscillator in two dimensions is

$$H = \omega(n_1 + n_2 + 1) \tag{1}$$

where, $n_i = a_i^\dagger a_i$, with $[a_i, a_j^\dagger] = \delta_{ij}$ and $[a_i, a_j] = 0$

(a) Work out the commutation relations of the set of operators $\{H, J_1, J_2, J_3\}$ where,

$$J_1 = \frac{1}{2}(a_2^\dagger a_1 + a_1^\dagger a_2), \quad J_2 = \frac{i}{2}(a_2^\dagger a_1 - a_1^\dagger a_2), \quad J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$$

(b) Show that $\mathbf{J^2} = J_1^2 + J_2^2 + J_3^2$ and J_3 form a complete commuting set and write down their eigenvectors and eigenvalues.

(c) Discuss the degeneracy of the spectrum and its splitting due to a small perturbation $\mathbf{V}.\mathbf{J}$ where $\mathbf{V}$ is a constant three component vector.

Problem 30 H.O in electric field (Written)

A charged particle is bound in a harmonic oscillator potential $V = 1/2Kx^2$. The system is placed in an external electric field E that is constant in space and time. Calculate the shift of the energy of the ground state to order E^2.

Problem 31 Rotation Matrices (Oral)

(a) Prove that $\frac{1}{\sqrt{2}}(1 + i\hat{\sigma}_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)

(b) Construct the matrix representation of $\hat{S}_z$ when the eigenkets of $\hat{S}_y$ are used as base vectors (i.e. in the basis where $\hat{S}_y$ is diagonal).

Problem 32 Spin-$\frac{1}{2}$ Particle in a Uniform Magnetic Field (Oral)

An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$ with eigenvalue $\frac{\hbar}{2}$, where $\boldsymbol{n}$ is a unit vector, lying in the xz-plane making an angle β with the z-axis.

(a) Obtain the probability for finding the electron in the $s_x = \frac{\hbar}{2}$ state as a function of time.

(b) Find the expectation value of $\hat{S}_x$ as a function of time.

(c) For your own peace of mind show that your answer makes good sense in the extreme cases (i) $\beta \to 0$ and (ii) $\beta \to \frac{\pi}{2}$.

Problem 33 Harmonic Oscillator (Written)

Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by

$$e^{i\hat{p}a/\hbar} |0\rangle , \tag{2}$$

where $\hat{p}$ is the momentum operator and a is some number with dimension of length. Using the Heisenberg picture, evaluate the expectation value $\langle \hat{x} \rangle$ for $t > 0$.