Problem 29 Perturbation (Oral)

The Hamiltonian for an isotropic harmonic oscillator in two dimensions is

\[ H = \omega (n_1 + n_2 + 1) \]  

where, \( n_i = a_i^\dagger a_i \), with \([a_i, a_j^\dagger] = \delta_{ij}\) and \([a_i, a_j] = 0\)

(a) Work out the commutation relations of the set of operators \( \{H, J_1, J_2, J_3\} \) where,

\[ J_1 = \frac{1}{2}(a_2^\dagger a_1 + a_1^\dagger a_2), \quad J_2 = \frac{i}{2}(a_2^\dagger a_1 - a_1^\dagger a_2), \quad J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \]

(b) Show that \( J^2 = J_1^2 + J_2^2 + J_3^2 \) and \( J_3 \) form a complete commuting set and write down their eigenvectors and eigenvalues.

(c) Discuss the degeneracy of the spectrum and its splitting due to a small perturbation \( V \cdot J \) where \( V \) is a constant three component vector.

Problem 30 H.O in electric field (Written)

A charged particle is bound in a harmonic oscillator potential \( V = 1/2Kx^2 \). The system is placed in an external electric field \( E \) that is constant in space and time. Calculate the shift of the energy of the ground state to order \( E^2 \).

Problem 31 Rotation Matrices (Oral)

(a) Prove that \( \frac{1}{\sqrt{2}}(1 + i\sigma_x) \) acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the \( x \)-axis by angle \(-\pi/2\). (The minus sign signifies that the rotation is clockwise.)

(b) Construct the matrix representation of \( \hat{S}_z \) when the eigenkets of \( \hat{S}_y \) are used as base vectors (i.e. in the basis where \( \hat{S}_y \) is diagonal).

Problem 32 Spin-\( \frac{1}{2} \) Particle in a Uniform Magnetic Field (Oral)

An electron is subject to a uniform, time-independent magnetic field of strength \( B \) in the positive \( z \)-direction. At \( t = 0 \) the electron is known to be in an eigenstate of \( \hat{S} \cdot n \) with eigenvalue \( \hbar/2 \), where \( n \) is a unit vector, lying in the \( xz \)-plane making an angle \( \beta \) with the \( z \)-axis.
(a) Obtain the probability for finding the electron in the $s_x = \frac{\hbar}{2}$ state as a function of time.

(b) Find the expectation value of $\hat{S}_x$ as a function of time.

(c) For your own peace of mind show that your answer makes good sense in the extreme cases 

$\beta \to 0 \quad \text{and} \quad \beta \to \frac{\pi}{2}$.

**Problem 33  Harmonic Oscillator (Written)**

Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by

$$e^{i\hat{p}a/\hbar} |0\rangle,$$

where $\hat{p}$ is the momentum operator and $a$ is some number with dimension of length. Using the Heisenberg picture, evaluate the expectation value $\langle \hat{x} \rangle$ for $t > 0$. 