
Problem Set 12 Due: 21/22 January 2016

Problem 37 Identical Particles inside Harmonic Oscillator Potential (Written)

Two identical bosons of mass m move in a one-dimensional harmonic oscillator potential $\hat{V} = \frac{1}{2}m\omega\hat{x}^2$. Suppose that they interact with each other via the potential

$$\hat{V}_{\text{int}}(x_1, x_2) = \alpha e^{-\beta(x_1-x_2)^2}, \tag{1}$$

where β is a positive constant. Determine the ground state energy of the system in terms of the interaction strength α using first-order time-independent perturbation theory.

Problem 38 Fermions Inside Infinite Square Well (Oral)

Two identical spin-$\frac{1}{2}$ fermions move in one dimension under the influence of the infinite-wall potential

$$V(x) = \begin{cases} \infty & |x| \geq a \\ 0 & |x| < a \end{cases} \tag{2}$$

(a) Write the ground-state wave function and the ground-state energy when the two particles are constrained to a spin-triplet state (*ortho* state).

(b) Repeat (a) when they are in a spin-singlet state (*para* state).

(c) Now suppose the two particles interact mutually via a very short-range attractive potential that can be approximated by

$$V_{\text{int}}(x_1, x_2) = -\lambda\delta(x_1 - x_2), \quad \lambda > 0. \tag{3}$$

Assuming that perturbation theory is valid, discuss semiquantitatively what happens to the energy levels obtained in (a) and (b).

Problem 39 Creation and Annihilation Operators (Oral)

If a unitary transformation is performed in the space of one-particle state vectors, then a unitary transformation is induced in the space of the operators themselves. Show that the creation and annihilation operators are then given by:

$$\begin{aligned} \hat{a}^\dagger(\boldsymbol{r}) &= \sum_n \langle n|\boldsymbol{r}\rangle\, \hat{a}^\dagger(n) \\ \hat{a}(\boldsymbol{r}) &= \sum_n \langle n|\boldsymbol{r}\rangle^*\, \hat{a}(n), \end{aligned} \tag{4}$$

with $\hat{a}^\dagger(\boldsymbol{r})\,|0\rangle = |\boldsymbol{r}\rangle$.

We say that $\hat{a}^\dagger(\boldsymbol{r})$ transforms like the ket $|\boldsymbol{r}\rangle$ whereas $\hat{a}(\boldsymbol{r})$ transforms like the bra $\langle\boldsymbol{r}|$. In position space it is customary to represent $\hat{a}^\dagger(\boldsymbol{r})$ by the operator $\hat{\psi}^\dagger(\boldsymbol{r})$ and $\hat{a}(\boldsymbol{r})$ by $\hat{\psi}(\boldsymbol{r})$.

Show that the fundamental commutation relations for bosonic and fermionic creation and annihilation operators remain the same, that is, show in this particular case that

$$\begin{aligned}\left[\hat{\psi}^\dagger(\boldsymbol{r}), \hat{\psi}^\dagger(\boldsymbol{r'})\right]_\pm &= 0\\ \left[\hat{\psi}(\boldsymbol{r}), \hat{\psi}(\boldsymbol{r'})\right]_\pm &= 0\\ \left[\hat{\psi}(\boldsymbol{r}), \hat{\psi}^\dagger(\boldsymbol{r'})\right]_\pm &= \delta^d(\boldsymbol{r}-\boldsymbol{r'})\end{aligned} \tag{5}$$

An operator like $\hat{\psi}(\boldsymbol{r})$, which depends on the position coordinates, is generally referred to as a **quantum field operator** or simply a **field**. They are extensively used in quantum field theories.