Advanced Quantum Theory **WS 2015/16s**

Problem Set 13 Due: 28/29 January 2016

Problem 41 Density Operator and Non-interacting Fermions (Oral)

Consider N non-interacting spin-$\frac{1}{2}$ fermions, all with spin-up, occupying the lowest available single particle orbitals in a one-dimensional box with impenetrable walls at $x = 0$ and $x = L$. The density operator is given by

$$\hat{\rho}(x) = \sum_{i=1}^{N} \delta(x - \hat{x}_i) \to \hat{\psi}^\dagger(x)\hat{\psi}(x) \tag{1}$$

in the field theory representation. Show that the average fermion density has N maxima and find their positions.

Note: $\sum_{j=1}^{N} \sin^2(j\theta) = \frac{2N+1}{4} - \frac{\sin[(2N+1)\theta]}{4\sin\theta}$

Problem 42 Number Operator in Second Quantization (Written)

For the normalized multi-particle state

$$|\psi\rangle = A\,|1,1,0,1,0,0,0...\rangle + B\,|1,1,1,1,0,0,0...\rangle\,, \tag{2}$$

written in occupation number representation, compute the expectation value of the particle number operator $\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k$ as a function of A.

Problem 43 Two-site Hubbard Model (Oral)

Consider the Hubbard Hamiltonian given by

$$\hat{H} = -t \sum_{\langle ij\rangle\sigma} \hat{a}_{i\sigma}^\dagger \hat{a}_{j\sigma} + U \sum_i \hat{a}_{i\uparrow}^\dagger \hat{a}_{i\uparrow} \hat{a}_{i\downarrow}^\dagger \hat{a}_{i\downarrow}, \tag{3}$$

where $\langle ij\rangle$ indicates that the sum is over nearest neighbors only and $\sigma = \uparrow, \downarrow$. Consider a lattice of only two sites (i.e. $i, j = 1, 2$).

(a) Assume there is only one spin-up electron occupying either site. Write down all possible configurations $\{|\sigma_1, \sigma_2\rangle\}$, determine the 2×2 Hubbard Hamiltonian in this basis and solve it.

(b) Now suppose there are two electrons in the system. If they have the same spin we obtain $E = 0$ (why?). If the electrons have different spins, there will be four different configurations. Write down the 4×4 Hubbard Hamiltonian in this basis and determine the energy eigenstates and eigenvalues.

Write down the eigenstates also in terms of creation operators.