Wiederholungsübung

1a) 3 erste Aufgaben : trivial

4. $\frac{d}{dx} \log\left(\frac{x}{\exp(x)-1}\right) = \frac{d}{dx}\left(\log x - \log(\exp(x)-1)\right) = \text{trivial}$

$\frac{1}{x} - \frac{\exp(x)}{\exp(x)-1}$

b) 1. Integral trivial.

2. Kettenregel invers oder Subst $dx\, 3x^2 = dx^3$

3. dto $dx\, 2x = dx^2$

4. 2 x partiell integrieren [Subst $\sin x = \frac{e^{ix}+e^{-ix}}{2}$ ist umständlich]

$\underset{u'\ v}{\int e^x \sin x\, dx} = \underset{u\ v}{e^x \sin x} - \int e^x \cos x\, dx$ $\Rightarrow \int dx\, e^x \sin x = \frac{1}{2} e^x (\sin x - \cos$

$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$

5. ~~für~~ Substitution $\underset{\overset{\shortparallel}{dt}}{dx \cos x} = d \sin x$, $\cos^2 x = 1 - \sin^2 x$, $t = \sin x$

Kürzen

$$\int dx \frac{\cos^3 x}{1-\sin x} = \int dt\, \frac{1-t^2}{1-t} = \int dt\,(1+t) = t + \frac{t^2}{2} = \sin x + \frac{1}{2}\sin^2 x$$

c) $\frac{d}{d\lambda} = \frac{\partial}{\partial\lambda} + \frac{\partial x}{\partial\lambda}\cdot\frac{\partial}{\partial x}$

damit $\frac{d}{d\lambda} \int_0^\lambda f(x,\lambda)\, dx = \int_0^\lambda \frac{\partial}{\partial\lambda} f(x,\lambda) dx + \frac{\partial x}{\partial\lambda}\frac{\partial}{\partial x} \underbrace{\int_{0=x_1}^{\lambda=x_2} f(x,\lambda) dx}_{\text{Abl. d. Integrals also } f(x,\lambda)}$.

$= \int_0^\lambda \frac{\partial}{\partial\lambda} f(x,\lambda) dx + f(x,\lambda) \frac{\partial x}{\partial\lambda}\Big|_{x_1}^{x_2}$

$$= \int_0^\lambda \frac{\partial}{\partial\lambda} f(x,\lambda) dx + f(\underset{\overset{\shortparallel}{\lambda}}{x_2}(\lambda),\lambda) \underset{=1}{\frac{\partial x_2}{\partial\lambda}} - \cancel{f(\underset{\overset{\shortparallel}{0}}{x_1(\lambda)},\lambda) \underset{\overset{\shortparallel}{0}}{\frac{\partial x_1}{\partial\lambda}}}$$

②

1c) $$\frac{d}{d\lambda}\int_0^{\lambda} dx\, \frac{e^{-\lambda x^2}-1}{x} = \int_0^{\lambda} dx\,(-x)^2\,\frac{e^{-\lambda x^2}-1}{x} + \frac{e^{-\lambda^3}-1}{\lambda}$$

$x_2(\lambda)=\lambda \quad x_1(\lambda)=0 \qquad \frac{\partial}{\partial\lambda}\left(\frac{e^{-\lambda x^2}}{x}-\frac{1}{x}\right)$

$$= \int_0^{\lambda} dx\,\left(-x e^{-\lambda x^2} \cancel{+x}\right) + \frac{e^{-\lambda^3}-1}{\lambda} = \frac{1}{2\lambda} e^{-\lambda x^2} \cancel{+\tfrac{1}{2}x^2}\Big|_0^{\lambda} + \frac{e^{-\lambda^3}-1}{\lambda}$$

Abl.

$$= \frac{e^{-\lambda^3}\cancel{+\lambda^3}-1}{2\lambda} + \frac{2e^{-\lambda^3}-2}{2\lambda} = \frac{3e^{-\lambda^3}\cancel{+\lambda^3}-3}{2\lambda} = 3\,\frac{e^{-\lambda^3}-1}{2}$$

2a) $$\nabla\left(\exp(x)\cos(y)+z\right) = \begin{pmatrix}\cos(y)\exp(x)\\ -\sin(y)\exp(x)\\ 1\end{pmatrix}$$

$$\Delta\left(\exp(x)\cos(y)+z\right) = \cos(y)\exp(x) - \cos(y)\exp(x) + 0 = 0$$

b) $$\frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial x} = \frac{\partial A_z}{\partial y} = \frac{\partial A_x}{\partial z} = 0$$

$$\frac{\partial A_x}{\partial y} = \left((a-3)+x(8x-4ax)\right)e^{-(x^2+y^2)} + \left(x(4-2a)+y(a-3)+xy(8x-4ax)\right)e^{-(x^2+y^2)}\cdot(-2$$

$$\frac{\partial A_x}{\partial x} = \left((3a-5)+y(8y-4ay)\right)e^{-(x^2+y^2)} + \left(x(3a-5)+y(4-2a)+xy(8x-4ax)\right)e^{-(x^2+y^2)}\cdot(-2$$

Rotations frei: $\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = 0$

Versuch $a=2$ passt nicht, da zwar viele Terme wegfallen, aber Konstante nicht

Koeffizientenvergleich nach Ausmultiplizieren : $a=1$

Damit:

$$A = \begin{pmatrix} 2x-2y+4x^2y\\ -2x+2y+4xy^2\\ 0\end{pmatrix} e^{-(x^2+y^2)} \quad \begin{matrix} = \frac{\partial\psi}{\partial x}\\ = \frac{\partial\psi}{\partial y}\\ = \end{matrix}$$

2b)

Skal Man sieht $\frac{\partial \psi}{\partial x} = (2x - 2y + 4x^2 y)\, e^{-(x^2+y^2)}$

Abl

bleibt: $(4x^2-2)\, y\, e^{-(x^2+y^2)}$ zu integrieren

Man sieht: $f = e^{-(x^2+y^2)}$ $f' = -2x\, e^{-(x^2+y^2)}$ $f'' = (4x^2-2)\, e^{-(x^2+y^2)}$

Also: $\frac{\partial \psi}{\partial x} = (4x^2-2)\, y\, e^{-(x^2+y^2)} + 2x\, e^{-(x^2+y^2)}$

$\psi = -2xy\, e^{-(x^2+y^2)} - e^{-(x^2+y^2)}$

$= -(2xy+1)\, e^{-(x^2+y^2)}$

$x, y$ symmetrisch, daher ergibt $\frac{\partial \psi}{\partial y}$ dasselbe.

$\frac{\partial \psi}{\partial z} = 0 \Rightarrow \psi = -(2xy+1)\, e^{-x^2+y^2} + C$

3a) $y = e^{kx}$ $y' = k e^{kx}$ $y'' = k^2 e^{kx}$ $y''' = k^3 e^{kx}$

$k^3 + 2k^2 + 2k = 0$ $k_0 = 0$, $k_{1,2} = -1 \pm i$ s.o.

$y = A + B e^{-x+ix} + C e^{-x-ix}$

$y' = A(-1+i)\, e^{-x+ix} + C(-1-i)\, e^{-x+ix}$

$y'' = B(-2i)\, e^{-x+ix} + C(2i)\, e^{-x+ix}$

3b) $x = 0$ $A + B + C = 0$ / $A = 2 - 2B \Rightarrow A = 0$

$B(-1+i) + C(-1-i) = -2$ $B(-2) = -2 \Rightarrow B = 1$

$B(-2i) + C(2i) = 0 \Rightarrow B = C$

$y = 2e^{-x}\cos x$

3c) $x=0$ $\quad A+B+C=2$

$B(-1+i)+C(-1-i) = -1+i$

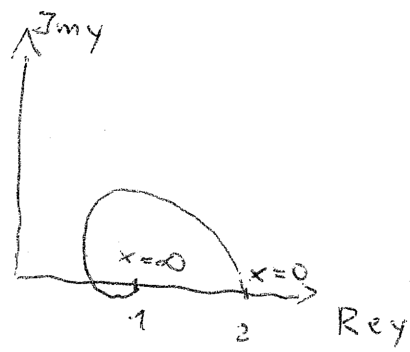
$x=\pi$ $\quad A-Be^{-\pi}-Ce^{-\pi} = 1-e^{-\pi}$

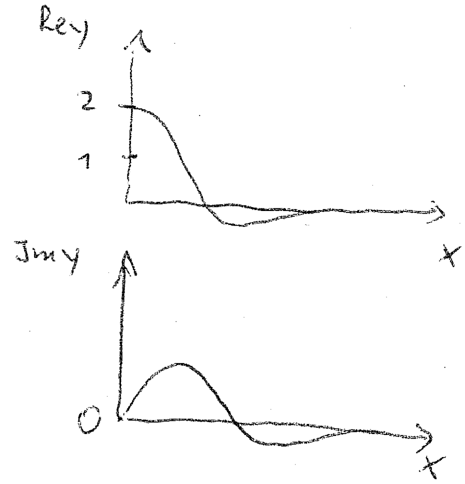
$A=1 \quad B+C=1 \quad B=1 \quad C=0$

$y = 1+e^{-x+ix}$

$\mathrm{Re}\, y = 1+e^{-x}\cos x$

$\mathrm{Im}\, y = e^{-x}\sin x$




3d) Ansatz: $y = a+bx+cx^2+\underline{\underline{dx^3}}$

$f\cdot x^2$

$y' = b+2cx+3dx^2$

$y'' = 2c+6dx$

$y''' = 6d$

$6d+4c+12dx+2b+4cx+6dx^2 = x^2$

$(6d+4c+2b)+(12d+4c)x+6dx^2 = x^2$

$\Rightarrow d=\frac{1}{6} \Rightarrow c=-\frac{1}{2} \Rightarrow b=\frac{1}{2} \quad a$ bel.

$y_{inh_1} = a+\frac{x}{2}-\frac{x^2}{2}+\frac{1}{6}x^3$

3d) Ansatz $\quad y = a\sin x + b\cos x$

f. $\cos x$ $\quad y' = a\cos x - b\sin x$

$y'' = -a\sin x - b\cos x$

$y''' = -a\cos x + b\sin x$

$\cos x$: $\quad -a - 2b + 2a = 1 \qquad a - 2b = 1 \qquad \Rightarrow \quad b = -\frac{2}{5}$

$\sin x$: $\quad b - 2a - 2b = 0 \qquad -b - 2a = 0 \qquad a = \frac{1}{5}$

$$y_{inh_2} = \frac{1}{5}\sin x - \frac{2}{5}\cos x$$

4a) Schnittpunkt: $\quad y_1(x) = y_2(x)$

$x^2 + 2x + 1 = 3x + 1 \quad \Rightarrow \quad x^2 = x \qquad x_{1,2} = 0, 1$

$$A = \int_0^1 dx \int_{x^2+2x+1}^{3x+1} dy = \int_0^1 (x - x^2)\, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{6}$$

$$A_x = \int_0^1 x\, dx \int_{x^2+2x+1}^{3x+1} dy = \int_0^1 x\, dx\, (x - x^2) = \int_0^1 dx\, (x^2 - x^3) = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{12}$$

$$\Rightarrow \quad S_x = \frac{A_x}{A} = \frac{1}{2}$$

$$A_y = \int_0^1 dx \int_{y_1}^{y_2} y\, dy = \int_0^1 dx\, \frac{1}{2} y^2 \Big|_{y_1}^{y_2} = \int_0^1 dx\, \frac{1}{2}(y_2^2 - y_1^2)$$

$y_2^2 = 9x^2 + 6x + 1 \qquad y_1^2 = x^4 + 4x^3 + 6x^2 + 4x + 1$

$y_2^2 - y_1^2 = 3x^2 + 2x - x^4 - 4x^3$

$$A_y = \frac{1}{2}\int_0^1 dx\, (3x^2 + 2x - x^4 - 4x^3) = \frac{1}{2}\left(x^3 + x - \frac{x^5}{5} - x^4\right)\Big|_0^1 = \frac{1}{2}\cdot\frac{4}{5} = \frac{2}{5}$$

$$\Rightarrow \quad S_y = \frac{A_y}{A} = \frac{12}{5}$$

4b) Rotationsparaboloid $z = 1 - x^2 - y^2$, $z = 0$

Polarkoordinaten; $z = 1 - r^2$ Rand $r = 1$

$$V = \iiint dx\,dy\,dz = \int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} r\,dr\,d\varphi\,dz = 2\pi \int_0^1 (r - r^3)\,dr$$

$$= 2\pi \left(\tfrac{1}{2}r^2 - \tfrac{1}{4}r^3\right) = \frac{\pi}{2}$$

Formel für Rotationskörper $V = \int \pi r^2 dz$ mit $r = \sqrt{1-z}$ ergibt dasselbe.

4c) Basis: $B = \int_0^{2\pi}\int_0^{\cancel{1}} r\,dr\,d\varphi = 2\pi \frac{1}{2} r^2 \Big|_0^1 = \pi$

Mantel: 3 Möglichkeiten

1. Formel f. Rotationskörper $M = 2\pi \int_0^1 r \underbrace{\sqrt{r'^2 + 1}}_{\sqrt{\frac{1}{4(1-z)} + 1}} dz$ $\quad r' = \frac{-1}{2\sqrt{1-z}}$

Lösung mit Subst. etc.

$$M = \frac{\pi}{4}\left(\sqrt{125} - 1\right)$$

2 Formel f. Oberflächen explizit

$$M = \iint \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy \qquad f_x = -2x, \quad f_y = -2y$$

mit ~~Kugel~~ Polar Koordinaten $M = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\,dr\,d\varphi$

$$= \pi \int_0^1 \sqrt{1+4r^2}\,dr^2 = \frac{\pi}{4}\sqrt{1+4r^2}^3 \Big|_0^1$$

$$= \frac{\pi}{4}\left(\sqrt{125} - 1\right)$$

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4c) Oberflächenintegral explizit

Parametrisierung: $x = r\cos\varphi$
$y = r\sin\varphi$
$z = 1 - r^2$

$$\frac{\partial\vec{r}}{\partial\varphi} = r\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} \quad \frac{\partial\vec{r}}{\partial r} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ -2r\end{pmatrix} \quad \frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\varphi} = \begin{pmatrix}2r^2\cos\varphi\\ 2r^2\sin\varphi\\ r\end{pmatrix}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\varphi}\right| = \sqrt{1+4r^2}\cdot r$$

$$M = \iint dS = \iint \left|\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\varphi}\right| dr\, d\varphi$$

kein r!

$$M = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\varphi = \frac{\pi}{4}\left(\sqrt{125}-1\right)$$

5a) $$\tilde{f}(n) = \frac{1}{L}\int_0^L e^{-ik_n x} e^x\, dx \qquad k_n = \frac{2\pi}{L}n \qquad L = 2\pi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{x-inx}\, dx = \frac{1}{2\pi}\frac{1}{1-in} e^{x-inx}\Big|_0^{2\pi} = \frac{1}{2\pi}\frac{1}{1-in}\left(e^{2\pi}-1\right)$$

da $\left(e^{-2\pi i}\right)^n = 1$

$$f(x) = \sum_{-\infty}^{\infty} \frac{e^{2\pi}-1}{2\pi}\cdot\frac{1}{1-in} e^{inx}$$

$$= \frac{e^{2\pi}-1}{2\pi}\sum_{-\infty}^{\infty}\frac{1+in}{1+n^2} e^{inx} = \frac{e^{2\pi}-1}{2\pi}\left(\sum_1^{\infty}\frac{2}{1+n^2}\underbrace{\frac{e^{inx}+e^{-inx}}{2}}_{\cos nx} + \right.$$

$$\left. -\sum_1^{\infty}\frac{2n\cancel{i}}{1+n^2}\underbrace{\frac{e^{inx}-e^{-inx}}{2i}}_{\sin nx} + \underbrace{1}_{n=0}\right)$$

5b) ⑧

$$h(x) = \sum_{-\infty}^{\infty} \hat{g}(n)\, e^{ik_n x} \underbrace{\frac{e^{i\frac{2\pi}{L}x} + e^{-i\frac{2\pi}{2}x}}{2}}_{\cos\frac{2\pi}{L}x} \qquad k = \frac{2\pi}{L} n$$

$$= \frac{1}{2} \sum_{-\infty}^{\infty} \hat{g}(\underset{m-1}{n})\, e^{i\frac{2\pi}{L}(\overset{m}{n+1})x} + \hat{g}(\underset{m'+1}{n})\, e^{i\frac{2\pi}{L}(\overset{m'}{n-1})x}$$

$m' \to m$

$$= \frac{1}{2} \sum_{-\infty}^{\infty} \hat{g}(m-1)\, e^{ik_m x} + \hat{g}(m+1)\, e^{ik_m x}$$

$$= \sum_{-\infty}^{\infty} \underbrace{\frac{1}{2}\left(\hat{g}(m-1) + \hat{g}(m+1)\right)}_{\hat{h}(m)} e^{ik_m x}$$

$$\hat{h}(n) = \frac{1}{2}\left(\hat{g}(n-1) + \hat{g}(n+1)\right)$$

zu 1c)

$$\frac{d}{d\lambda} \int_{x_1(\lambda)}^{x_2(\lambda)} dx\, f(x,\lambda) = \frac{d}{d\lambda}\left[F(x_2(\lambda),\lambda) - F(x_1(\lambda),\lambda)\right]$$

$$= \frac{d}{dx_2} F(x_2(\lambda),\lambda) \frac{dx_2}{d\lambda}(\lambda) - \frac{dF_1}{dx_1}(x_1(\lambda),\lambda) \frac{dx_1}{dx_1}(\lambda)$$

$$+ \underbrace{\frac{\partial}{\partial\lambda} F(x_2,\lambda) - \frac{\partial}{\partial\lambda} F(x_1,\lambda)}_{\int_{x_1}^{x_2} \frac{\partial}{\partial\lambda} f(x,\lambda)\, dx}$$